Math 31 - Homework 7

Due Friday, August 17

1. Let R be a ring, and suppose that I and J are ideals in R. Prove that $I \cap J$ is an ideal in R.

2. Let R be a commutative ring, and fix $a \in R$. Define the **annihilator** of a to be the set

$$\text{Ann}(a) = \{x \in R : xa = 0\}.$$

Prove that $\text{Ann}(a)$ is an ideal of R.

3. Let R be a commutative ring. An element $a \in R$ is said to be **nilpotent** if there is a positive integer n such that $a^n = 0$. The set

$$\text{Nil}(R) = \{a \in R : a \text{ is nilpotent}\}$$

is called the **nilradical** of R. Prove that the nilradical is an ideal of R. [**Hint:** You may need to use the fact that the usual binomial theorem holds in a commutative ring. That is, if $a, b \in R$ and $n \in \mathbb{Z}^+$, then

$$(a+b)^n = \sum_{k=0}^{n} a^{n-k} b^k.$$

This should help with checking that $\text{Nil}(R)$ is closed under addition.]

4. Let R and S be two rings with identity, and let 1_R and 1_S denote the multiplicative identities of R and S, respectively. Let $\varphi : R \to S$ be a nonzero ring homomorphism. (That is, φ does not map every element of R to 0.)

(a) Show that if $\varphi(1_R) \neq 1_S$, then $\varphi(1_R)$ must be a zero divisor in S. Conclude that if S is an integral domain, then $\varphi(1_R) = 1_S$.

(b) Prove that if $\varphi(1_R) = 1_S$ and $u \in R$ is a unit, then $\varphi(u)$ is a unit in S and

$$\varphi(u^{-1}) = \varphi(u)^{-1}.$$

5. Let R be a commutative ring with identity.

(a) Fix $\alpha \in R$. Define the **evaluation homomorphism at** α to be the map $\text{ev}_\alpha : R[x] \to R$ given by: if $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is in $R[x]$, then

$$\text{ev}_\alpha(p) = a_n \alpha^n + \cdots + a_1 \alpha + a_0.$$

Show that ev_α is indeed a ring homomorphism.

(b) Determine the kernel of ev_α.

(c) Suppose now that $R[x]$ is a PID. Show that the kernel of ev_α is a maximal ideal, and conclude that R must be a field in this case.

6. Determine whether each of the following polynomials is irreducible over the given field.

(a) $3x^4 + 5x^3 + 50x + 15$ over $\mathbb{Q}$.

(b) $x^2 + 7$ over $\mathbb{Q}$.

(c) $x^2 + 7$ over $\mathbb{C}$.